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Courant algebroid and Lie bialgebroid contractions*

José F Cariñena[1], **Janusz Grabowski**[2] **and Giuseppe Marmo**[3,4]

[1] Depto. Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain
[2] Mathematical Institute, Polish Academy of Sciences, ul. Śniadeckich 8, PO Box 21, 00-956 Warszawa, Poland
[3] Dipartimento di Scienze Fisiche, Università Federico II di Napoli, Napoli, Italy
[4] INFN, Sezione di Napoli, Complesso Universitario di Monte Sant'Angelo, Via Cintia, 80125 Napoli, Italy

E-mail: jfc@posta.unizar.es, jagrab@impan.gov.pl and marmo@na.infn.it

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Abstract
Contractions of Leibniz algebras and Courant algebroids by means of (1, 1)-tensors are introduced and studied. An appropriate version of Nijenhuis tensors leads to natural deformations of Dirac structures and Lie bialgebroids. One recovers presymplectic-Nijenhuis structures, Poisson–Nijenhuis structures and triangular Lie bialgebroids as particular examples.

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1. Introduction

This paper is a natural continuation of our previous work [CGMb], where contractions and Nijenhuis tensors have been studied for algebraic operations of arbitrary type on sections of vector bundles. Recall that a *Nijenhuis tensor* N for a bilinear operation '$\circ$' on sections of a vector bundle A over M is a (1, 1)-tensor $N \in \mathrm{Sec}(A \otimes A^*)$ viewed as vector bundle morphism $N : A \to A$ (or the corresponding $C^\infty(M)$-linear map $N : \mathrm{Sec}(A) \to \mathrm{Sec}(A)$ on sections) such that its *Nijenhuis torsion*

$$T_N(X, Y) = N(X) \circ N(Y) - N(X \circ_N Y) \tag{1}$$

vanishes, where '$\circ_N$' is the contracted product

$$X \circ_N Y = N(X) \circ Y + X \circ N(Y) - N(X \circ Y). \tag{2}$$

The theory of Nijenhuis tensors for Lie algebra brackets goes back to a concept of contractions of Lie algebras introduced by Saletan [Sa]. Nijenhuis tensors for Lie algebroids and Nijenhuis

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tensors on Poisson manifolds were studied in [MM, KSM] and in a number of following papers. In [CGMa] the authors of this paper developed the theory of Nijenhuis tensors for associative products, and in [CGMb]—for arbitrary algebraic operations.

One can apply directly the procedures from [CGMb] to Leibniz algebras. The vanishing of the Nijenhuis torsion implies that the contracted product '$\circ_N$' is again a Leibniz product. However, as we will see in the example of the Courant product on $\mathsf{T}M \oplus \mathsf{T}^*M$ (in its Leibniz version) the vanishing of the Nijenhuis torsion is a restrictive assumption too. To get that '$\circ_N$' is Leibniz it is sufficient to require that T_N is a Leibniz 2-cocycle. We will refer to such tensors N as to *weak Nijenhuis tensors* for Leibniz algebras. Since the use of weak Nijenhuis tensors does not lead to contractions in the strict sense (they do not come from a limit procedure), one should rather call '$\circ_N$' a *deformed product* in this case. So the convention throughout this paper is that we use the word 'contraction' heuristically, thinking just on a procedure of passing from a product '$\circ$' to the product '$\circ_N$' for a specifically chosen $(1, 1)$-tensor N.

To introduce a notion of a Lie bialgebroid contraction, we use the concept of *Courant algebroid* [LWX] in its Leibniz version. Since the Courant algebroid is not only a Leibniz product but also a non-degenerate pairing with certain consistency conditions with the Leibniz product, we check what property of N ensures the consistency conditions being satisfied also for '$\circ_N$'. It turns out that it is sufficient to assume that $N + N^* = \lambda I$, $\lambda \in \mathbb{R}$, where N^* is dual to N with respect to the pairing; we will call such tensors *paired*. Thus, paired and weak Niejnhuis tensors on Courant algebroids give rise to deformed Courant algebroids.

There is a straightforward but very useful generalization of the concept of the Nijenhuis tensor. Suppose that L is a subbundle of A whose sections are closed with respect to the operation '$\circ$', i.e. they form a subalgebra in $(\mathrm{Sec}(A), \circ)$. If $\mathrm{Sec}(L)$ is closed also for '$\circ_N$' and the torsion T_N vanishes on L, i.e. $T_N(X, Y) = 0$ for all $X, Y \in \mathrm{Sec}(L)$, we will refer to N as to an *outer Nijenhuis tensor* for $(L, \circ)$. This concept seems to be the right tool in contracting *Dirac structures*, i.e. subbundles of Courant algebroids which are maximal isotropic and closed with respect to the product. In this approach, a *Dirac–Nijenhuis* structure is an outer Nijenhuis tensor N for a Dirac subbundle L such that '$\circ_N$' is skew-symmetric on $\mathrm{Sec}(L)$, so that '$\circ_N$' is a deformed Lie algebroid bracket on L. A particular case is when N is a weak Nijenhuis and paired tensor on a Courant algebroid which is an outer Nijenhuis tensor for L. In this case the subbundle L is a Dirac structure for the deformed Courant algebroid product '$\circ_N$'.

Finally, Lie bialgebroids are known as complementary to each other Dirac subbundles (structures) E_1, E_2 in a Courant algebroid A, $E_1 \oplus E_2 = A$. It is therefore completely natural to call by *Lie bialgebroid–Nijenhuis structure* any tensor N on A which yields Dirac–Nijenhuis structures for both: E_1 and E_2. The deformed bracket restricted to E_1 and E_2 gives two Lie algebroid brackets and the consistency condition (N is paired) is satisfied, so we get a new Lie bialgebroid. It is interesting that, associated with particular contractions, we recover presymplectic-Nijenhuis and Poisson–Nijenhuis structures (cf [MM, KSM]). Since the latter play a prominent role in the theory of integrable systems, this discovery supports once more the conviction on the importance of bi- or double-structures such as Lie bialgebras, Manin triples, Lie bialgebroids, Courant algebroids, etc, in complete integrability. Note that a close relation of Poisson–Nijenhuis structures with Lie bialgebroids was first observed by Kosmann-Schwarzbach [KS] (see also [GUa]).

2. Contractions of Leibniz algebras and the Courant bracket

The language of *Leibniz algebras* is very useful in description of *Lie bialgebroids* in the sense of Mackenzie and Xu [MX]. In [CGMb], the theory of contractions has been developed for binary operations of arbitrary type, so that all this general theory of contractions can be directly applied to Leibniz products (or brackets) on sections of a vector bundle A, in particular for Courant algebroids and Lie bialgebras.

Definition 1. *A Leibniz product (bracket) on a vector space $\mathcal{A}$ is a bilinear operation '$\circ$' satisfying the Jacobi identity*

$$(X \circ Y) \circ Z = X \circ (Y \circ Z) - Y \circ (X \circ Z) \tag{3}$$

for all $X, Y, Z \in \mathcal{A}$. The space $\mathcal{A}$ which is equipped with a Leibniz product we call a Leibniz algebra.

Remark that Leibniz algebras as non-skew-symmetric generalizations of Lie algebras were first studied by Loday [Lo] (they are called sometimes *Loday algebras*), and a major part of (co)homology theory of Lie algebras was generalized to Leibniz algebras. Let now '$\circ$' be a Leibniz product on the space $\mathcal{A} = \mathrm{Sec}(A)$ of sections of a vector bundle A over M which is local, i.e. which is locally defined by a bidifferential operator, and let $N : A \to A$ be a $(1, 1)$-tensor over A. According to the general scheme in [CGMb], if the Nijenhuis torsion (1) vanishes, the contracted product (2) is a Leibniz product which is *compatible* with the original one, i.e. $X \circ_N Y + \lambda X \circ Y$ is a Leibniz product for any $\lambda \in \mathbb{R}$. However, we can have the same under much weaker conditions.

Lemma 1. *The products '$\circ_N$' and '$\circ$' are always compatible in the sense that*

$$\begin{aligned}(X \circ_N Y) \circ Z - X \circ_N (Y \circ Z) + Y \circ_N (X \circ Z)\\ + (X \circ Y) \circ_N Z - X \circ (Y \circ_N Z) + Y \circ (X \circ_N Z) = 0.\end{aligned} \tag{4}$$

Proof. Direct computations with the use of the Jacobi identity (3) for '$\circ$'. □

Theorem 1. *The contracted product (2) is still Leibniz if and only if the Nijenhuis torsion (1) is a 2-cocycle with respect to the Leibniz cohomology operator, i.e.*

$$\begin{aligned}(\delta T_N)(X, Y, Z) = T_N(X, Y \circ Z) - T_N(X \circ Y, Z) - T_N(Y, X \circ Z)\\ - T_N(X, Y) \circ Z + X \circ T_N(Y, Z) - Y \circ T_N(X, Z) = 0.\end{aligned} \tag{5}$$

In this case '$\circ_N$' and '$\circ$' are compatible Leibniz products.

Proof. One proves that

$$(X \circ_N Y) \circ_N Z - X \circ_N (Y \circ_N Z) + Y \circ_N (X \circ_N Z) = (\delta T_N)(X, Y, Z) \tag{6}$$

by direct computations using the Jacobi identity for '$\circ$' and the compatibility condition (4). In the case when '$\circ_N$' is a Leibniz product, the Jacobi identity for the product $X \circ_N Y + \lambda X \circ Y$ reduces to (4). □

The tensor N we will call a *Nijenhuis tensor* (for the Leibniz algebra $\mathcal{A}$) if the Nijenhuis torsion T_N vanishes and a *weak Nijenhuis tensor* if the Nijenhuis torsion T_N is a Leibniz 2-cocycle. In both cases the contracted product '$\circ_N$' is Leibniz and is compatible with the original one.

An interesting example of a Leibniz product is the following version of the *Courant bracket* on sections $X + \xi$ of the bundle $\mathsf{T}M \oplus \mathsf{T}^*M$:

$$(X + \xi) \circ (Y + \eta) = [X, Y] + (\mathcal{L}_X \eta - \mathrm{i}_Y \,\mathrm{d}\xi). \tag{7}$$

This is an example of a *Courant algebroid* associated with the trivial Lie bialgebroid $((\mathsf{T}M, [\cdot, \cdot]), (\mathsf{T}^*M, 0))$ with the standard Lie algebroid structure on $\mathsf{T}M$ and the trivial one on T^*M (cf [LWX, Ro]). If we have a Nijenhuis tensor N_0 for $\mathsf{T}M$, we can contract the standard bracket of vector fields to a Lie algebroid bracket $[X, Y]_{N_0} = [N_0X, Y] + [X, N_0Y] - N_0[X, Y]$

(cf [KSM, CGMb]). We obtain another trivial Lie bialgebroid $((\mathsf{T}M, [\cdot,\cdot]_{N_0}), (\mathsf{T}^*M, 0))$ with the corresponding Courant bracket

$$(X+\xi)\circ^{N_0}(Y+\eta) = [X,Y]_{N_0} + \left(\mathcal{L}_X^{N_0}\eta - \mathrm{i}_Y\,\mathrm{d}^{N_0}\xi\right) \tag{8}$$

where d^{N_0} and $\mathcal{L}^{N_0}$ denote the de Rham differential and the Lie derivative, respectively, associated with the Lie algebroid $(\mathsf{T}M, [\cdot,\cdot]_{N_0})$. It is a matter of standard calculations to show that $\mathrm{d}^{N_0} = \mathrm{i}_{N_0}\mathrm{d} - \mathrm{d}i_{N_0}$, where i_{N_0} is the derivation of the algebra of differential forms generated by N_0 (see [KSM, GUa]). We may as well speak of the product (8) purely formally, not even assuming that N_0 is a Nijenhuis tensor, and get the following:

Theorem 2. *The product '$\circ^{N_0}$' defined by (8) is actually the contracted product '$\circ_N$' with $N(X+\xi) = N_0X - {}^tN_0\xi$, where ${}^tN_0 : \mathsf{T}^*M \to \mathsf{T}^*M$ is the dual map: $\langle X, {}^tN_0\xi\rangle = \langle N_0X, \xi\rangle$, i.e.*

$$\begin{aligned}(X+\xi)\circ^{N_0}(Y+\eta) &= [X,Y]_{N_0} + (N_0X)\circ\eta - X\circ({}^tN_0\eta) + {}^tN_0(X\circ\eta)\\ &\quad - ({}^tN_0\xi)\circ Y + \xi\circ(N_0Y) + {}^tN_0(\xi\circ Y).\end{aligned} \tag{9}$$

Proof. We have

$$\begin{aligned}\left\langle\mathcal{L}_X^{N_0}\eta, Y\right\rangle &= (N_0X)\langle\eta,Y\rangle - \left\langle\eta,[X,Y]_{N_0}\right\rangle\\ &= (N_0X)\langle\eta,Y\rangle - \langle\eta,[N_0X,Y]+[X,N_0Y]-N_0[X,Y]\rangle\\ &= \left\langle\mathcal{L}_{N_0X}\eta + {}^tN_0(\mathcal{L}_X\eta) - \mathcal{L}_X({}^tN_0\eta), Y\right\rangle.\end{aligned}$$

The rest can be proved analogously. □

Since, for N being Nijenhuis, the contracted bracket '$\circ^{N_0} = \circ_N$' is clearly a Leibniz bracket, the tensor N is automatically weak Nijenhuis in this case. On the other hand, what is rather unexpected, the tensor N is a Nijenhuis tensor for the Courant bracket (7) only in very particular and rare cases. Namely, we have the following.

Theorem 3. *For a Nijenhuis tensor $N_0 : \mathsf{T}M \to \mathsf{T}M$ on a connected manifold M, the tensor $N : \mathsf{T}M \oplus \mathsf{T}^*M \to \mathsf{T}M \oplus \mathsf{T}^*M$, $N(X+\xi) = N_0X - {}^tN_0\xi$, is a Nijenhuis tensor for the Courant bracket (7) if and only if $N_0^2 = \lambda I$ for certain $\lambda \in \mathbb{R}$.*

Proof. Since T_N vanishes on $\mathsf{T}M$ and on T^*M separately, the vanishing of T_N on $\mathsf{T}M \oplus \mathsf{T}^*M$ is equivalent to the system of identities

$${}^tN_0\mathcal{L}_X^{N_0}\eta = \mathcal{L}_{N_0X}({}^tN_0\eta) \tag{10}$$

$${}^tN_0\mathrm{i}_Y\,\mathrm{d}^{N_0}\xi = \mathrm{i}_{N_0Y}\,\mathrm{d}({}^tN_0\xi) \tag{11}$$

for all $X, Y \in \mathrm{Sec}(\mathsf{T}M)$ and $\eta, \xi \in \mathrm{Sec}(\mathsf{T}^*M)$. The first one is equivalent to

$$N_0[N_0X, Y] = [X, N_0Y]_{N_0}$$

for all $X, Y \in \mathrm{Sec}(\mathsf{T}M)$ and, due to vanishing of the Nijenhuis torsion of N_0, to

$$N_0^2[X,Y] = \left[X, N_0^2Y\right].$$

Since (11) in the presence of (10) can be replaced by

$$({}^tN_0)^2\,\mathrm{d}\langle Y,\xi\rangle = \mathrm{d}\langle Y, ({}^tN_0)^2\xi\rangle$$

the proof follows by the following lemma. □

Lemma 2. *If a* (1, 1)*-tensor* $K : \mathsf{T}M \to \mathsf{T}M$ *on a connected manifold* M *commutes with the adjoint action of vector fields, i.e.*

$$K[X, Y] = [X, KY] \tag{12}$$

for all $X, Y \in \mathrm{Sec}(\mathsf{T}M)$*, then* $K = \lambda I$ *for certain* $\lambda \in \mathbb{R}$.

Proof. In local coordinates (x^i) and the corresponding coordinate vector fields (∂_i), we can write $K(\partial_j) = K^i_j(x)\partial_i$ and, according to (12),

$$\left[\partial_k, K^i_j(x)\partial_i\right] = \frac{\partial K^i_j}{\partial x^k}(x)\partial_i = 0$$

for all k, j (we use the Einstein's summation convention), so the coefficients $K^i_j(x) = K^i_j$ are constant. Hence, (12) applied to $X = x^1\partial_k$, $Y = \partial_1$, gives $K^i_k = \delta^i_k K^1_1$, i.e. $K = \lambda I$, where $\lambda = K^1_1$. This locally defined constant λ serves for the whole M, since M is connected. □

Note that (1, 1)-tensors $N_0 : \mathsf{T}M \to \mathsf{T}M$ with $N_0^2 = \lambda I$ and constant rank are *special* in the terminology of [BC]. They are proportional to such tensors with $\lambda = 0, \pm 1$. The case $\lambda = -1$ is the case of an *almost complex structure*, $\lambda = 1$ is the case of an *almost product structure*, and $\lambda = 0$ is the case of an *almost tangent structure*. If N_0 is additionally a Nijenhuis tensor, we deal with a *complex, product, and tangent structure*, respectively, cf [BC].

Corollary 1. *A Nijenhuis tensor* $N_0 : \mathsf{T}M \to \mathsf{T}M$ *gives rise to a Nijenhuis tensor* $N = N_0 \oplus (-{}^tN_0) : \mathsf{T}M \oplus \mathsf{T}^*M \to \mathsf{T}M \oplus \mathsf{T}^*M$ *for the standard Courant bracket if and only if* N_0 *is proportional to a complex, a product or a tangent structure on* M.

Such structures are extremely interesting from the geometric point of view. However, from an algebraic point of view, the contracted Courant brackets for complex and product structures are isomorphic with the original Courant bracket. To enrich the family of contracted brackets we will work also with weaker versions of Nijenhuis tensors. This approach will be systematically developed in the next sections for the general Courant algebroids.

3. Contractions of Courant algebroids. Dirac–Nijenhuis structures

A Courant algebroid is not only a Courant product '$\circ$' on sections of a vector bundle A but also a nondegenerate symmetric pairing $\langle\cdot,\cdot\rangle$ on A with certain consistency relations. The general contraction procedure in such a case is obvious: we contract the product and check if the consistency conditions with other structures are still satisfied. If this is the case, we call such contraction the contraction of the whole structure and the corresponding Nijenhuis tensor we call the Nijenhuis tensor for the global structure.

Let us recall briefly the structure of a Courant algebroid. We will use the Leibniz bracket version of the Courant product (bracket) presented in [Ro] with some simplifications (cf [GM, definition 1], [KS1, definition 2.1] and [Uch]). Thus the 'compressed' definition is as follows:

Definition 2. *A Courant algebroid is a vector bundle* $\tau : A \to M$ *with a Leibniz product (bracket) '*$\circ$*' on* $\mathrm{Sec}(A)$*, a vector bundle map (over the identity)* $\rho : A \to \mathsf{T}M$ *and a nondegenerate symmetric bilinear form* $\langle\cdot,\cdot\rangle$ *on* A *satisfying the identities*

$$\rho(X)\langle Y, Y\rangle = 2\langle X, Y \circ Y\rangle \tag{13}$$

$$\rho(X)\langle Y, Y\rangle = 2\langle X \circ Y, Y\rangle. \tag{14}$$

Note that (13) is equivalent to

$$\rho(X)\langle Y, Z\rangle = \langle X, Y \circ Z + Z \circ Y\rangle. \tag{15}$$

Similarly, (14) easily implies the invariance of the pairing $\langle\cdot,\cdot\rangle$ with respect to the left multiplication

$$\rho(X)\langle Y, Z\rangle = \langle X \circ Y, Z\rangle + \langle Y, X \circ Z\rangle \tag{16}$$

and that ρ is the anchor map for the left multiplication:

$$X \circ (fY) = fX \circ Y + \rho(X)(f)Y. \tag{17}$$

Assume now that N is a (1, 1)-tensor on A and consider the 'contracted' product (2). We do not assume that N is Nijenhuis at the moment. Exactly as in the classical case of a Lie algebroid contraction [CGMb, lemma 2], we have the anchor $\rho_N = \rho \circ N$ for the contracted multiplication

$$X \circ_N (fY) = f(X \circ_N Y) + \rho(NX)(f)Y. \tag{18}$$

Now, let us check under what conditions the identities (13) and (14) are still satisfied for '$\circ_N$'. Let N^* be the adjoint of N with respect to the pairing:

$$\langle NX, Y\rangle = \langle X, N^*Y\rangle$$

and let $\Delta = N + N^*$. Using the invariance (14), we get easily

$$\begin{aligned}\langle X \circ_N Y, Z\rangle &= \langle NX \circ Y + X \circ NY - N(X \circ Y), Z\rangle \\ &= \rho(NX)\langle Y, Z\rangle - \langle Y, NX \circ Z\rangle + \rho(X)\langle NY, Z\rangle - \langle NY, X \circ Z\rangle - \langle X \circ Y, N^*Z\rangle \\ &= \rho(NX)\langle Y, Z\rangle - \langle Y, NX \circ Z\rangle + \langle Y, N^*(X \circ Z)\rangle + \langle Y, X \circ N^*Z\rangle\end{aligned}$$

which equals $\rho(NX)\langle Y, Z\rangle - \langle Y, X \circ_N Z\rangle$ if and only if

$$\langle Y, X \circ \Delta Z - \Delta(X \circ Z)\rangle = 0$$

for all X, Y, Z, i.e. if and only if Δ commutes with the left multiplication

$$X \circ \Delta Z - \Delta(X \circ Z) = 0. \tag{19}$$

Thus (19) is equivalent to the invariance of the pairing with respect to '$\circ_N$':

$$\rho_N(X)\langle Y, Z\rangle = \langle X \circ_N Y, Z\rangle + \langle Y, X \circ_N Z\rangle.$$

Similarly, checking (13) for '$\circ_N$', we get

$$\begin{aligned}\langle X, Y \circ_N Y\rangle &= \langle X, NY \circ Y + Y \circ NY - N(Y \circ Y)\rangle \\ &= \rho(X)\langle Y, NY\rangle - \langle N^*X, Y \circ Y\rangle \\ &= \tfrac{1}{2}\rho(X)\langle Y, \Delta Y\rangle - \tfrac{1}{2}\rho(N^*X)\langle Y, Y\rangle\end{aligned}$$

which equals $\frac{1}{2}\rho(NX)\langle Y, Y\rangle$ if and only if

$$\rho(X)\langle Y, \Delta Y\rangle = \rho(\Delta X)\langle Y, Y\rangle.$$

The latter can be rewritten in the form

$$\langle X, Y \circ \Delta Y + \Delta Y \circ Y\rangle = 2\langle \Delta X, Y \circ Y\rangle$$

or

$$Y \circ \Delta Y + \Delta Y \circ Y = 2\Delta(Y \circ Y).$$

Using (19) we get finally the condition

$$\Delta(Y \circ Y) = \Delta Y \circ Y. \tag{20}$$

Theorem 4. *If $N : A \to A$ is a $(1, 1)$-tensor on a Courant algebroid, then the contracted product (2) is compatible with the symmetric pairing $\langle\cdot,\cdot\rangle$ of the Courant algebroid, in the sense that (13) and (14) are satisfied for '$\circ_N$' and ρ_N, if and only if*

$$X \circ (N + N^*)Y = (N + N^*)(X \circ Y) \qquad \text{and} \qquad (N + N^*)(Y \circ Y) = (N + N^*)Y \circ Y$$

for all sections X, Y of A.

Of course, how restrictive the above conditions are, depends on 'how irreducible' is the Courant product. However, there is one case which works for any Courant algebroid, namely the case $N + N^* = \lambda I$, $\lambda \in \mathbb{R}$.

Definition 3. *A $(1, 1)$-tensor on a Courant algebroid we call paired if $N + N^* = \lambda I$ for some $\lambda \in \mathbb{R}$. A paired (weak) Nijenhuis tensor we call (weak) Courant–Nijenhuis tensor.*

Thus weak Courant–Nijenhuis tensors give rise to contractions, or better to say—deformations, of Courant algebroids. Note, however, that the structure of a Courant algebroid is extremely rigid and that there are very few true Courant–Nijenhuis tensors. First, observe that N is a Courant–Nijenhuis tensor if and only $N - \frac{\lambda}{2}I$ is Courant–Nijenhuis (cf [CGMb, theorem 8]), so we can always reduce to the case when $N + N^* = 0$. We have the following generalization of theorem 3:

Theorem 5. *If N is a Courant–Nijenhuis tensor with $N + N^* = 0$, then N^2 commutes with the left multiplication:*

$$X \circ N^2 Y = N^2(X \circ Y)$$

and $N^2(Y \circ Y) = (N^2 Y) \circ Y$.

Proof. Using $N^* = -N$ and the invariance of the pairing, we get

$$\langle N(X \circ_N Y), Z\rangle = -\langle X \circ_N Y, NZ\rangle = -\rho(NX)\langle Y, NZ\rangle + \langle Y, X \circ_N NZ\rangle \tag{21}$$

and

$$\langle NX \circ NY, Z\rangle = \rho(NX)\langle NY, Z\rangle + \langle Y, N(NX \circ Z)\rangle \tag{22}$$

so N is Nijenhuis implies that the rhs of (21) and (22) are equal, i.e.

$$X \circ_N NZ - N(NX \circ Z) = 0. \tag{23}$$

But the lhs of (23) is

$$NX \circ NZ - N(X \circ_N Z) - N^2(X \circ Z) + X \circ N^2 Z$$

and vanishing of the Nijenhuis torsion implies $N^2(X \circ Z) = X \circ N^2 Z$. One proves the second identity analogously, see the proof of (20). □

Remark. The above property of N is a strong restriction indeed. We know already that in the case of the standard Courant bracket this implies that N^2 is proportional to the identity (cf theorem 3). One can see this problem as the problem of small intersection of the properties: being paired and being Nijenhuis. Indeed, exactly as in [CGMb], any Leibniz–Nijenhuis tensor N gives rise to a whole hierarchy of compatible Leibniz structures and Leibniz–Nijenhuis tensors of the form N^k while N^2, for a paired N, is usually not paired. Thus the concept of a hierarchy for Courant algebroid should be reworked. For example, one can consider only odd powers or add an additional 'twist' to all powers of N. We will not discuss this problem in this paper working, in principle, with generalized versions of Nijenhuis tensors. For example, one can weaken the assumption for a paired tensor N to determine a proper contraction assuming just that the tensor N is weak Nijenhuis, i.e. we will admit weak Courant–Nijenhuis tensors as well. For a weak Courant–Nijenhuis tensor N on a Courant algebroid A, the product '$\circ_N$' defines another Courant algebroid product with respect to the same pairing and the anchor ρ_N, and '$\circ_N$' is compatible with '$\circ$', i.e. $N + \lambda I$ is a one-parameter family of weak Courant–Nijenhuis tensors (cf theorem 1).

Let now L be a *Dirac structure* in the Courant algebroid A, i.e. let L be a subbundle which is maximal isotropic and closed with respect to the Leibniz product '$\circ$'.

Definition 4. *The pair* (L, N) *we call a Dirac–Nijenhuis structure if N is a* $(1, 1)$*-tensor in A such that the deformed product '*$\circ_N$*' is closed and skew-symmetric on L and the Nijenhuis torsion* T_N *vanishes on L.*

Theorem 6. *Let L be a Dirac structure in the Courant algebroid* $(A, \circ, \langle\cdot, \cdot\rangle)$

(a) If a paired $(1, 1)$*-tensor N on A is an outer Nijenhuis tensor for L then* (L, N) *is a Dirac–Nijenhuis structure.*
(b) If (L, N) *is a Dirac–Nijenhuis structure, then L is a Lie algebroid with respect to the product '*$\circ_N$*' and* $N(X \circ_N Y) = NX \circ NY$ *for* $X, Y \in \mathrm{Sec}(L)$.

Proof. (a) Since N is paired, the consistency conditions (13), (14) are satisfied for '$\circ_N$' that implies the skew-symmetry of '$\circ_N$' on any isotropic subbundle.

(b) The deformed product '$\circ_N$' has the anchor $\rho \circ N$ and, due to (6) the vanishing of the Nijenhuis torsion on L implies that '$\circ_N$' satisfies the Jacobi identity (3) on L. □

Examples. Our Courant algebroid will be $A = \mathsf{T}M \oplus \mathsf{T}^*M$ with the standard Courant product (bracket)

$$(X + \xi) \circ (Y + \eta) = [X, Y] + (\mathcal{L}_X \eta - \mathrm{i}_Y \, \mathrm{d}\xi).$$

1. Let L be the Dirac subbundle in A associated with a closed 2-form Ω, i.e. section of L is of the form $X + \Omega X$ for X being vector fields on M. The fact that Ω is closed can be expressed in terms of the Courant product '$\circ$' by the identity

$$\mathrm{d}\Omega(X, Y, \cdot) = X \circ \Omega Y + \Omega X \circ X - \Omega[X, Y] = 0. \tag{24}$$

We will refer to any closed 2-form as to a *presymplectic structure*. Note however that, strictly speaking, a presymplectic structure is often understood as a closed 2-form of constant rank. We do not make any assumption on the rank of Ω in this paper. Let N_0 be a $(1, 1)$-tensor on $\mathsf{T}M$, and let $N(X + \xi) = N_0 X$ be an associated $(1, 1)$-tensor on A. Let us check under what conditions (L, N) is a Dirac–Nijenhuis structure. First of all, L should be closed with respect to the deformed bracket '$\circ_N$'. Since, as easily seen,

$$(X + \Omega X) \circ_N (Y + \Omega Y) = [X, Y]_{N_0} + N_0 X \circ \Omega Y + \Omega X \circ N_0 Y \tag{25}$$

this condition is equivalent to

$$N_0 X \circ \Omega Y + \Omega X \circ N_0 Y - \Omega[X, Y]_{N_0} = 0 \tag{26}$$

which can be rewritten in the form

$$\begin{aligned}(N_0 X \circ \Omega Y + \Omega N_0 X \circ Y - \Omega[N_0 X, Y]) + (\Omega X \circ N_0 Y + X \circ \Omega N_0 Y - \Omega[X, N_0 Y])\\ - (\Omega N_0 X \circ Y + X \circ \Omega N_0 Y - \Omega N_0[X, Y])\\ = \mathrm{d}\Omega(N_0 X, Y, \cdot) + \mathrm{d}\Omega(X, N_0 Y, \cdot) - \mathrm{d}(\Omega N_0)(X, Y, \cdot)\\ = -\mathrm{d}(\Omega N_0)(X, Y, \cdot) = 0\end{aligned}$$

where we have denoted

$$\mathrm{d}(\Omega N_0)(X, Y, \cdot) = \Omega N_0 X \circ Y + X \circ \Omega N_0 Y - \Omega N_0[X, Y]$$

independently on the skew-symmetry of ΩN_0. But the condition

$$\mathrm{d}(\Omega N_0)(X, Y, \cdot) = 0$$

implies immediately that ΩN_0 is skew-symmetric, i.e. $\Omega N_0 = {}^t N_0 \Omega$. Indeed,

$$\mathrm{d}(\Omega N_0)(X, X, \cdot) = \mathrm{d}(\Omega N_0(X, X)) = 0$$

for all vector fields X, so $\Omega N_0(X, X) = 0$ for all vector fields X and ΩN_0 is skew-symmetric. Thus, L is closed with respect to '$\circ_N$' if and only if ΩN_0 is skew-symmetric and $\mathrm{d}(\Omega N_0) = 0$. In this case

$$(X + \Omega X) \circ_N (Y + \Omega Y) = [X, Y]_{N_0} + \Omega[X, Y]_{N_0}.$$

Finally, the Nijenhuis torsion of N vanishes on L if and only if

$$N((X + \Omega X) \circ_N (Y + \Omega Y)) = N_0([X, Y]_{N_0}) = N(X + \Omega X) \circ N(Y + \Omega Y) = [N_0 X, N_0 Y]$$

i.e. N_0 is a classical Nijenhuis tensor. This structure is known as *presymplectic-Nijenhuis structure* (called in [MM] ΩN-*structure*), so that (L, N) as above is a Dirac–Nijenhuis structure if and only if (Ω, N_0) is a presymplectic-Nijenhuis structure.

2. Let L be as above but take the (1, 1)-tensor on A of the triangular form: $N(X + \xi) = \Lambda\xi$, for some $\Lambda : \mathsf{T}^*M \to \mathsf{T}M$. The deformed product on L reads

$$(X + \Omega X) \circ_N (Y + \Omega Y) = [X, Y]_{N_0} + N_0 X \circ \Omega Y + \Omega X \circ N_0 Y$$

where $N_0 = \Lambda\Omega$, so it exactly like (25). We conclude that (L, N) is Dirac–Nijenhuis in this case if and only if $\Lambda\Omega$ is a Niejnhuis tensor, $\Omega\Lambda\Omega$ is skew-symmetric, and $\mathrm{d}(\Omega\Lambda\Omega) = 0$. In [MM] such structures are called $\Lambda\Omega$-*structures*.

3. Let now the Dirac subbundle L of A will be associated with a Poisson tensor Λ, i.e. sections of L are of the form $\Lambda\xi + \xi$ for ξ being 1-forms, and the Lie algebroid bracket reads

$$(\Lambda\xi + \xi) \circ (\Lambda\eta + \eta) = [\Lambda\xi, \Lambda\eta] + [\xi, \eta]^\Lambda$$

where

$$[\xi, \eta]^\Lambda = \Lambda\xi \circ \eta + \xi \circ \Lambda\eta$$

is the well-known bracket of 1-forms associated with the Poisson tensor Λ. Put $N(X + \xi) = N_0 X$ for some (1, 1)-tensor N_0 on $\mathsf{T}M$. Since

$$(\Lambda\xi + \xi) \circ_N (\Lambda\eta + \eta) = [\Lambda\xi, \Lambda\eta]_{N_0} + N_0\Lambda\xi \circ \eta + \xi \circ N_0\Lambda\eta$$

requiring the skew-symmetry of this product, we immediately get that $N_0\Lambda$ must be skew-symmetric, i.e.

$$N_0\Lambda = \Lambda {}^t N_0 \tag{27}$$

and that

$$(\Lambda\xi + \xi) \circ_N (\Lambda\eta + \eta) = [\Lambda\xi, \Lambda\eta]_{N_0} + [\xi, \eta]^{N_0\Lambda}.$$

Using (27) we can rewrite $[\Lambda\xi, \Lambda\eta]_{N_0}$ as $\Lambda\big([\xi, \eta]^\Lambda_{{}^t N_0}\big)$, where $[\cdot, \cdot]^\Lambda_{{}^t N_0}$ is the deformation of $[\cdot, \cdot]^\Lambda$ by ${}^t N_0$, so that the condition that '$\circ_N$' is closed on L can be written as

$$\Lambda\big([\xi, \eta]^\Lambda_{{}^t N_0} - [\xi, \eta]^{N_0\Lambda}\big) = 0. \tag{28}$$

The vanishing of the Nijenhuis torsion of N on L takes the form

$$[\Lambda\xi, \Lambda\eta]_{N_0} = [N_0\Lambda\xi, N_0\Lambda\eta]. \tag{29}$$

This simply means that the Nijenhuis torsion of N_0 vanishes on the image of Λ. The conditions (27), (28) and (29) form a weaker version of what is called a *Poisson–Nijenhuis structure* (ΛN_0*-structure* in the terminology of [MM]) for which the conditions read: $N_0\Lambda$ is skew-symmetric, N_0 is Nijenhuis and (instead of (28))

$$[\xi,\eta]^{\Lambda}_{{}^tN_0} - [\xi,\eta]^{N_0\Lambda} = 0$$

(cf [MM, KSM]).

4. Contractions of Lie bialgebroids

The origin of the concept of Courant algebroid [LWX] was an attempt to obtain double objects for Lie bialgebroids in the sense of Mackenzie and Xu [MX]. Suppose now that both E and E^* are Lie algebroids over M with brackets $[\cdot,\cdot]_E$ and $[\cdot,\cdot]_{E^*}$, anchors a and a_*, respectively. Let d_E (resp., d_{E^*}) be the de Rham differential and $\mathcal{L}^E$ (resp., $\mathcal{L}^{E^*}$) be the corresponding Lie derivative associated with the Lie algebroid structure on E (resp., E^*). We will denote sections of E by capitals and sections of E^* by greek letters, and we will often suppress the indices in the brackets, de Rham differentials and Lie derivatives if it will be clear from the context which Lie algebroid they come from.

On $A = E \oplus E^*$ there is a natural symmetric nondegenerate bilinear form:

$$\langle X+\xi, Y+\eta\rangle = \langle \xi, Y\rangle + \langle \eta, X\rangle. \tag{30}$$

It is well known (cf [Ro, example 2.6.7]) that the bundle A with the symmetric pairing $\langle\cdot,\cdot\rangle$, the anchor $\rho = a + a_*$ and the product

$$(X+\xi)\circ(Y+\eta) = ([X,Y] + \mathcal{L}_\xi Y - \mathrm{i}_\eta\,\mathrm{d}X) + ([\xi,\eta] + \mathcal{L}_X\eta - \mathrm{i}_Y\,\mathrm{d}\xi) \tag{31}$$

is a Courant algebroid if and only if the pair (E, E^*) is a Lie bialgebroid. The subbundles E and E^* are in this case *Dirac subbundles*, i.e. maximal isotropic with respect to the symmetric pairing and closed with respect to the Courant bracket, transversal to each other. Conversely (see [LWX]), if L_1 and L_2 are Dirac subbundles transversal to each other of a Courant algebroid A, then (L_1, L_2) is a Lie bialgebroid, where the brackets and anchors are just restrictions of the corresponding structures of the Courant algebroid and L_2 is considered as the dual bundle of L_1 under the Courant pairing. The Courant product is then of the form (31) and it is completely determined by the Lie algebroid structures on E and E^*. We have, namely

$$\langle X\circ\eta, Y\rangle = a(X)\langle\eta, Y\rangle - \langle\eta, X\circ Y\rangle \tag{32}$$

$$\langle X\circ\eta, \xi\rangle = -a_*(\eta)\langle X,\xi\rangle + a(\xi)\langle X,\eta\rangle + \langle X, \eta\circ\xi\rangle. \tag{33}$$

This nice characterization of Lie bialgebroids allows us to define naturally a concept of contraction of a Lie bialgebroid.

Definition 5. *Let* (E, E^*) *be a Lie bialgebroid, and N be a paired* $(1,1)$*-tensor on the Courant algebroid* $(A = E\oplus E^*, \rho, \circ, \langle\cdot,\cdot\rangle)$*. The triple* (E, E^*, N) *we call Lie bialgebroid–Nijenhuis structure if N is an outer Nijenhuis tensor for both E and* E^**.*

Theorem 7. *If* (E, E^*, N) *is a Lie bialgebroid–Nijenhuis structure, then* $((E, (\circ_N)_{|E}), (E^*, (\circ_N)_{|E^*}))$ *is again a Lie bialgebroid. Moreover, N is a weak Courant–Nijenhuis tensor in the Courant algebroid* $E\oplus E^*$ *and* $\circ_N$ *coincides with the Courant product* $\circ^N$ *associated with the contracted Lie bialgebroid* $((E, (\circ_N)_{|E}), (E^*, (\circ_N)_{|E^*}))$*.*

Proof. The contractions $((E, (\circ_N)_{|E})$ and $(E^*, (\circ_N)_{|E^*}))$ are clearly Lie algebroid structures on E and E^*, respectively. The tensor N being paired respects the consistency conditions, so that $((E, (\circ_N)_{|E}), (E^*, (\circ_N)_{|E^*}))$ is a Lie bialgebroid and $\circ_N$ is a new Courant bracket, so N is a weak Courant–Nijenhuis tensor. The product $\circ_N$ must coincide with $\circ^N$, since the Courant bracket in $E \oplus E^*$ is uniquely determined by the Lie algebroid structures in E and E^*. □

Let us look closer at the contractions of Lie bialgebroids. First of all, the splitting $A = E \oplus E^*$ induces the matrix form of N:

$$N = \begin{pmatrix} N_E & \Lambda \\ \Omega & N_{E^*} \end{pmatrix} \tag{34}$$

where N_E and N_{E^*} act on E and E^*, respectively, and $\Lambda : E^* \to E$, $\Omega : E \to E^*$. The tensor N being paired satisfies $N + N^* = \lambda I$. For $X, Y \in \mathrm{Sec}(E)$ we have

$$\langle N_E(X) + \Omega(X), Y\rangle = \langle NX, Y\rangle = \langle X, \lambda Y - N_E(Y) - \Omega(Y)\rangle$$

so

$$\langle \Omega(X), Y\rangle = -\langle X, \Omega(Y)\rangle \tag{35}$$

i.e. Ω is skew-symmetric and can be understood as a section of $\bigwedge^2 E^*$. We will refer to Ω as to a 2-form. Similarly, Λ is a section of $\bigwedge^2 E$, referred to as a bivector field. Finally, it is easy to see that

$$N_E + {}^tN_{E^*} = \lambda I_E \tag{36}$$

where the tensor ${}^tN_{E^*}$ represents the map ${}^tN_{E^*} : E \to E$ dual to $N_{E^*} : E^* \to E^*$. Conversely, if Λ, Ω are skew-symmetric and N_E and N_{E^*} satisfy (36), then (34) is a paired tensor.

Clearly, $X \circ_N Y = X \circ_{N_E} Y + X \circ_\Omega Y$. Using the obvious notation $(X + \xi)_E = X$ and $(X + \xi)_{E^*} = \xi$, we get

$$X \circ_N Y = X \circ_{N_E} Y + (\Omega X \circ Y + X \circ \Omega Y)_E + (\Omega X \circ Y + X \circ \Omega Y - \Omega(X \circ Y))_{E^*}.$$

Thus the condition that E is closed with respect to $\circ_N$ reads

$$(\Omega X \circ Y + X \circ \Omega Y - \Omega(X \circ Y))_{E^*} = 0. \tag{37}$$

But

$$(\Omega X \circ Y + X \circ \Omega Y - \Omega(X \circ Y))_{E^*} = \mathcal{L}_X(\Omega Y) - \mathrm{i}_Y\, \mathrm{d}(\Omega X) - \Omega([X, Y]_E) = \mathrm{d}\Omega(X, Y, \cdot)$$

so that E is closed with the bracket $\circ_N$ if and only if Ω is a closed 2-form. The analogous statement is, of course, valid for E^*. Note that we will denote the lhs of (37) also $\mathrm{d}\Omega$ even in the case when Ω is not skew-symmetric. Of course, in this case $\mathrm{d}\Omega$ has a meaning as a map and not as a 3-form. Similarly, let us see that

$$(\Omega X \circ Y + X \circ \Omega Y)_E = \mathcal{L}_{\Omega X}(Y) - \mathcal{L}_{\Omega Y}(X) + \mathrm{d}_{E^*}(\Omega(X, Y)) = [X, Y]^\Omega \tag{38}$$

is the standard form of the bracket $[\cdot, \cdot]^\Omega$ defined on E by the 'bivector field' $\Omega \in \mathrm{Sec}(\bigwedge^2 E^*)$. In the case when Ω is a 'Poisson tensor', i.e. the Schouten bracket $[\Omega, \Omega]_{E^*}$ vanishes, the bracket $[\cdot, \cdot]^\Omega$ is known to be a Lie algebroid bracket. We will denote the rhs of (38) by $[X, Y]^\Omega$ also when Ω is not Poisson and not even skew-symmetric. We get the following:

Theorem 8. *Let (E, E^*) be a Lie bialgebroid, and let N be a paired tensor of the form (34) on the Courant algebroid $E \oplus E^*$. Then the subbundle E (resp., E^*) is closed with respect to the contracted bracket '$\circ_N$' if and only if Ω (resp., Λ) is a closed 2-form with respect to the Lie algebroid structure on E (resp., E^*), i.e. $\Omega \in \mathrm{Sec}(\bigwedge^2 E^*)$ and $\mathrm{d}_E\Omega = 0$ (resp., $\Lambda \in \mathrm{Sec}(\bigwedge^2 E)$ and $\mathrm{d}_{E^*}\Lambda = 0$). In this case the bracket '$\circ_N$' on E (resp., on E^*) is of the form $X \circ_N Y = [X, Y]_{N_E} + [X, Y]^\Omega$ (resp., $\eta \circ_N \xi = [\eta, \xi]_{N_{E^*}} + [\eta, \xi]^\Lambda$).*

Let us now check what is meant by the vanishing of the Nijenhuis torsion on E (and, by duality, on E^*). Comparing the parts in E and E^*, we get two equations

$$N_E([X,Y]_{N_E} + [X,Y]^{\Omega}) = [N_E X, N_E Y]_E + (\Omega X \circ N_E Y + N_E X \circ \Omega Y)_E \tag{39}$$

$$\Omega([X,Y]_{N_E} + [X,Y]^{\Omega}) = [\Omega X, \Omega Y]_{E^*} + (\Omega X \circ N_E Y + N_E X \circ \Omega Y)_{E^*}. \tag{40}$$

They can be rewritten in the form

$$T_{N_E}(X,Y) + [X,Y]^{\Omega}_{N_E} - [X,Y]^{\Omega N_E} = 0$$
$$[\Omega X, \Omega Y]_{E^*} - \Omega([X,Y]^{\Omega}) - \mathrm{d}(\Omega N_E)(X,Y,\cdot) = 0$$

where T_{N_E} is the Nijenhuis torsion of N_E with respect to the Lie algebroid bracket on E, the bracket $[\cdot,\cdot]^{\Omega}_{N_E}$ is the contraction of $[\cdot,\cdot]^{\Omega}$ with respect to N_E, the bracket $[\cdot,\cdot]^{\Omega N_E}$ is given by (38) but for (possibly non-skew-symmetric) ΩN_E and the exterior derivative $\mathrm{d}(\Omega N_E)$ is given by (37) but for (possibly non-skew-symmetric) ΩN_E. Thus we get the following:

Theorem 9. *The matrix (34) acting on $A = E \oplus E^*$ gives rise to a Lie bialgebroid–Nijenhuis structure if and only if the following conditions are satisfied:*

(1) $N_E + {}^t N_{E^*} = \lambda I_E$ for some $\lambda \in \mathbb{R}$;
(2) Ω and Λ are skew-symmetric and closed: $\mathrm{d}_E(\Omega) = 0$, $\mathrm{d}_{E^*}(\Lambda) = 0$;
(3) The following identities hold:

$$T_{N_E}(X,Y) + [X,Y]^{\Omega}_{N_E} - [X,Y]^{\Omega N_E} = 0 \tag{41}$$

$$[\Omega X, \Omega Y]_{E^*} - \Omega([X,Y]^{\Omega}) - \mathrm{d}_E(\Omega N_E)(X,Y,\cdot) = 0 \tag{42}$$

$$T_{N_{E^*}}(\eta,\xi) + [\eta,\xi]^{\Lambda}_{N_{E^*}} - [\eta,\xi]^{\Lambda N_{E^*}} = 0 \tag{43}$$

$$[\Lambda\eta, \Lambda\xi]_E - \Lambda([\eta,\xi]^{\Lambda}) - \mathrm{d}_{E^*}(\Lambda N_{E^*})(\eta,\xi,\cdot) = 0. \tag{44}$$

Remark. The tensors ΩN_E and ΛN_{E^*} need not be skew symmetric in general. However, if the Lie algebroid structure on E is (locally) non-degenerate in the sense that the anchor map, thus d_E, is (locally) non-zero, then they have to be skew-symmetric. Indeed, (42) implies that $\mathrm{d}_E(\Omega N_E)(X,X,\cdot) = 0$. But $\mathrm{d}_E(\Omega N_E)(X,X,\cdot) = \mathrm{d}_E(\Omega(X,X))$, so $\Omega(X,X) = 0$ and Ω is (locally) skew-symmetric. Similarly, (43) implies that $[\eta,\eta]^{\Lambda N_{E^*}} = 0$. But $[\eta,\eta]^{\Lambda N_{E^*}} = \mathrm{d}_E(\Lambda N_{E^*}(\eta,\eta))$, so $\Lambda N_{E^*}(\eta,\eta) = 0$ and ΛN_{E^*} is (locally) skew-symmetric.

Now consider the trivial Lie bialgebroid $(E, E^*) = (\mathsf{T}M, \mathsf{T}^*M)$ with the standard bracket of vector fields on $\mathsf{T}M$ and the trivial bracket on T^*M. Then $\mathrm{d}_{E^*} = 0$, $\mathcal{L}^{E^*} = 0$, the brackets generated by Ω and ΩN_E are trivial and the above conditions for the matrix

$$N = \begin{pmatrix} \frac{\lambda}{2} I + N_0 & \Lambda \\ \Omega & \frac{\lambda}{2} I - {}^t N_0 \end{pmatrix} \tag{45}$$

where Ω is a closed 2-form (a presymplectic structure) and Λ is a bivector field, reduce to

$$T_{N_0} = 0 \tag{46}$$

$$\mathrm{d}(\Omega N_0) = 0 \tag{47}$$

$$[\eta,\xi]^{\Lambda}_{{}^t N_0} - [\eta,\xi]^{\Lambda {}^t N_0} = 0 \tag{48}$$

$$[\Lambda\eta, \Lambda\xi]_E - \Lambda([\eta,\xi]^{\Lambda}) = 0. \tag{49}$$

Note that, according to the above remark, in this case ΩN_0 and $\Lambda^t N_0$ are skew-symmetric automatically. The equation (46) means that N_0 is a (standard) Nijenhuis tensor which, together with the presymplectic form Ω, constitutes a presymplectic-Nijenhuis structure (ΩN-structure) [MM] according to (47). The identity (49) means that Λ is a Poisson tensor and (48) is a compatibility condition with N_0 which says that we deal with a Poisson–Nijenhuis structure (cf [MM, KSM, GUa]). Thus we get the following.

Theorem 10. *The Lie bialgebroid–Nijenhuis tensors* $N : \mathsf{T}M \oplus \mathsf{T}^*M \to \mathsf{T}M \oplus \mathsf{T}^*M$ *for the standard Courant bracket (7) for the trivial Lie bialgebroid* $(\mathsf{T}M, \mathsf{T}^*M)$ *are precisely of the form*

$$N = \begin{pmatrix} \frac{\lambda}{2} I + N_0 & \Lambda \\ \Omega & \frac{\lambda}{2} I - {}^t N_0 \end{pmatrix} \tag{50}$$

where N_0 *is a Nijenhuis tensor,* (N_0, Ω) *is a presymplectic-Nijenhuis structure and* (N_0, Λ) *is a Poisson–Nijenhuis structure.*

Remark that for a general trivial Lie bialgebroid $((E, [\cdot, \cdot]), (E^*, 0))$ the contracted Lie bialgebroid associated with the triangular matrix

$$N = \begin{pmatrix} I & \Lambda \\ 0 & I \end{pmatrix} \tag{51}$$

is the *triangular Lie bialgebroid* associated with the 'Poisson tensor' Λ in the standard terminology. Note also that the use of outer Nijenhuis tensors puts a flavour of interaction with the ambient bundle to the contracted products. For example, the above triangular tensor deforms the trivial bracket in E^* into a possibly non-trivial bracket $[\cdot, \cdot]^\Lambda$ induced by the Lie algebroid structure in E.

5. Concluding remarks

We have developed the idea of contractions of Courant algebroids, Dirac structures and Lie bialgebroids as a procedure of deforming such structures by means of appropriate Nijenhuis tensors. The standard Nijenhuis tensor approach turned out to be too restrictive, so we had to deal with tensors whose Nijenhuis torsion vanishes only on a subbundle in question. We should stress that this idea is of a conceptual nature rather than an *ad hoc* choice of definitions. The naturality of our approach is supported by the fact that we can recover basic examples of the interplay between the fundamental tensors in pairwise dual bundles, such as Poisson–Nijenhuis structures, presymplectic-Nijenhuis structures, etc, which have been studied in mathematics and physics in the context of integrability. We hope to find direct applications of our formalism in bi-Hamiltonian formalism and integrability in forthcoming papers.

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